Dynamical Systems (ME221)

Week 06: Laplace Transform and Transfer Function

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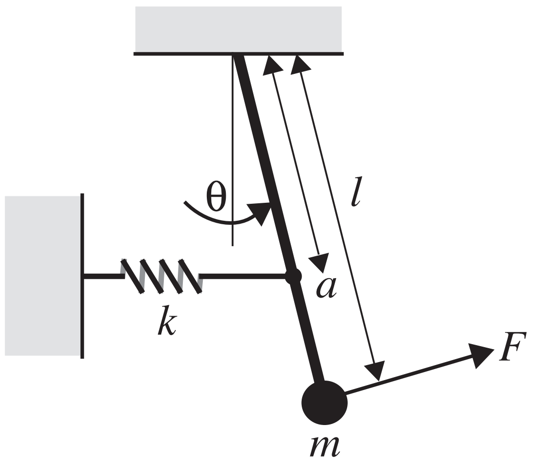
Lagrangian Formulation and External Forces



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

$$Q_i = \sum_j \vec{F}_j \cdot \frac{\partial \vec{r}_j}{\partial q_i}$$

- Non-conservative virtual work
  - Forces that cannot be derived from a potential function V
  - Externally applied forces, Q_i, fall into this category

- As with all the generalized quantities, pay attention to the interpretation
- If the generalized coordinate represents an angle, the generalized force will be a **torque**

Lecture Overview

- Laplace Transform
- Transfer function

Next Week:

- Inverse Laplace Transform

Laplace Transform

- For a given function $f(t)$ with $f(t) = 0$ for $t < 0$, Laplace transform of this function is defined as follows:

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

s is a complex variable

$$s = a + jb$$

$$f(t) \xrightarrow{\mathcal{L}} F(s)$$

- From differential equations to algebraic equations

Laplace Transform of Common Functions

- **Exponential function**

$$f(t) = \begin{cases} 0, & t < 0 \\ Ae^{-\alpha t}, & t \geq 0 \end{cases}$$

$$F(s) = \mathcal{L}[Ae^{-\alpha t}] = \int_0^\infty Ae^{-\alpha t}e^{-st}dt = A\int_0^\infty e^{-(\alpha+s)t}dt = \frac{A}{s+\alpha}$$

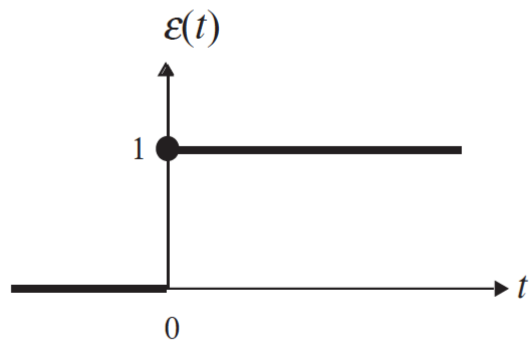
- We assumed that the real part of s is greater than -α (the abscissa of convergence), so that the integral converges.

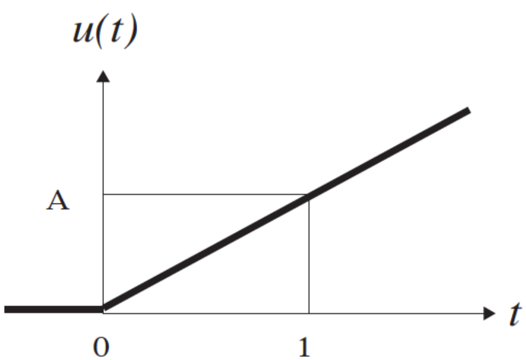
Laplace Transform of Common Functions

- **Unit Step function**

$$\varepsilon(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

$$\mathcal{L}[\varepsilon(t)] = \int_0^\infty 1 e^{-st} \mathrm{d}t = \frac{1}{s}$$





- **Ramp function**

$$u(t) = \begin{cases} 0 & t < 0 \\ At & t \geq 0 \end{cases}$$

$$U(s) = \int_0^\infty Ate^{-st}\mathrm{d}t = A\left\{\left[-t\frac{1}{s}e^{-st}\right]\Bigg|_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st}\mathrm{d}t\right\} = \frac{A}{s^2}$$

Integration by Parts

$$\int_a^b u(x)v'(x)\,dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)dx$$

$$= u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x)\,dx$$

Laplace Transform of Common Functions

- **Sinusoidal function**

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \qquad \sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

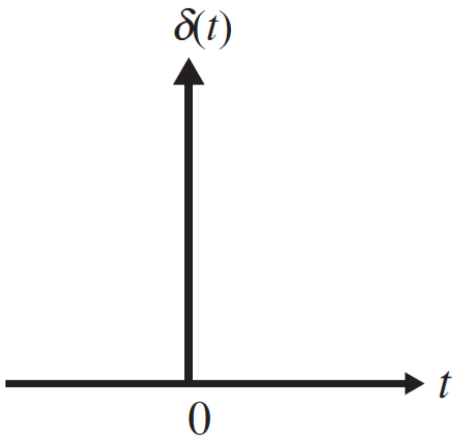
$$u(t) = \begin{cases} 0 & t < 0 \\ A\sin(\omega t) & t \geq 0 \end{cases} \xrightarrow{\mathcal{L}} U(s) = \frac{A\omega}{s^2 + \omega^2}$$

$$u(t) = \begin{cases} 0 & t < 0 \\ A\cos(\omega t) & t \geq 0 \end{cases} \xrightarrow{\mathcal{L}} U(s) = \frac{As}{s^2 + \omega^2}$$

Laplace Transform of Common Functions

- **Impulse function**

$$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1$$



$$\mathcal{L}[\delta(t)] = \int_{0}^{\infty} \delta(t)e^{-\mathrm{st}}\mathrm{d}t = \int_{0}^{0+} \delta(t)e^{0}\mathrm{d}t + \int_{0+}^{\infty} 0e^{-\mathrm{st}}\mathrm{d}t = 1$$

Laplace Transform of the Derivative of a Function

$$\mathcal{L}\left[\frac{\mathrm{d}}{\mathrm{d}t} f(t)\right] = \int_0^\infty \left[\frac{\mathrm{d}}{\mathrm{d}t} f(t)\right] e^{-\mathrm{st}} \mathrm{d}t$$

- **Integration by parts**

$$\int_0^\infty \left[\frac{\mathrm{d}}{\mathrm{d}t} f(t)\right] e^{-\mathrm{st}} \mathrm{d}t = [f(t)e^{-\mathrm{st}}]\Big|_0^\infty - \int_0^\infty f(t)(-s)e^{-\mathrm{st}}\mathrm{d}t$$

$$= -f(0) + s\int_0^\infty f(t)e^{-\mathrm{st}}\mathrm{d}t = sF(s) - f(0)$$

Laplace Transform of the Derivative of a Function

- **Higher order derivatives**

$$\mathcal{L}\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(t)\right] = s^2F(s) - sf(0) - \frac{\mathrm{d}}{\mathrm{d}t} f(0)$$

$$\mathcal{L}\left[\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t)\right] = s^nF(s) - s^{n-1}f(0) - \ldots - \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} f(0)$$

Laplace Transform of the Integral of a Function

$$\mathcal{L}\left[\int_0^t f(\tau)\mathrm{d}\tau\right] = \int_0^\infty \left[\int_0^t f(\tau)\mathrm{d}\tau\right] e^{-st}\mathrm{d}t$$

- **Integration by parts**

$$\mathcal{L}\left[\int_0^t f(\tau)\mathrm{d}\tau\right] = \left[\int_0^t f(\tau)\mathrm{d}\tau\right]\left(-\frac{1}{s}\right)e^{-st}\Bigg|_0^\infty - \int_0^\infty f(t)\left(-\frac{1}{s}\right)e^{-st}\mathrm{d}t$$

$$= -\frac{1}{s}\left[e^{-\infty}\int_0^\infty f(\tau)\mathrm{d}\tau - e^0\int_0^0 f(\tau)\mathrm{d}\tau\right] + \frac{1}{s}\int_0^\infty f(t)e^{-st}\mathrm{d}t = \frac{F(s)}{s}$$

Properties of Laplace Transform

- **The Linearity Property**

$$\mathcal{L}[f_1(t)] = F_1(s)$$

$$\mathcal{L}[f_2(t)] = F_2(s)$$

$$\mathcal{L}[c_1 f_1(t) + c_2 f_2(t)] = c_1 F_1(s) + c_2 F_2(s)$$

Properties of Laplace Transform

- **Time shift**

$$\mathcal{L}[\varepsilon(t-\tau)f(t-\tau)] = \int_0^\infty \varepsilon(t-\tau)f(t-\tau)e^{-\mathrm{st}}\mathrm{d}t = \int_\tau^\infty f(t-\tau)e^{-\mathrm{st}}\mathrm{d}t$$

substitute

$$v = t-\tau \qquad \mathcal{L}[\varepsilon(t-\tau)f(t-\tau)] = e^{-\mathrm{s}\tau}\int_0^\infty f(v)e^{-\mathrm{s}v}\mathrm{d}v = e^{-\mathrm{s}\tau}F(s)$$

- **Shift along the s-axis**

$$\mathcal{L}[e^{-\lambda \mathrm{t}}f(t)] = \int_0^\infty e^{-\lambda \mathrm{t}}f(t)e^{-\mathrm{st}}\mathrm{d}t = \int_0^\infty f(t)e^{-(\mathrm{s}+\lambda)\mathrm{t}}\mathrm{d}t = F(s+\lambda)$$

Properties of Laplace Transform

- **Multiplication by t**

$$F(s) = \mathcal{L}[tx(t)\,] = -\frac{dX(s)}{ds}$$

$$\mathcal{L}[t^{n} f(t)] \;=\; (-1)^{n} \frac{d^{n}}{ds^{n}} F(s)$$

- **Multiplication by 1/t**

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_{s}^{\infty} f(u)du \qquad \text{assuming that} \quad \mathcal{L}\left[\frac{f(t)}{t}\right] \to 0 \; as \; s \to \infty$$

Properties of Laplace Transform

- **Final value theorem**

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt = sF(s) - f(0) \qquad (1)$$

$$\lim_{s\to 0} \int_0^\infty e^{-st} f'(t)dt = \lim_{s\to 0} \lim_{T\to\infty} \int_0^T e^{-st} f'(t)dt = \lim_{s\to 0} \lim_{T\to\infty} \{e^{-sT} f(T) - f(0)\}$$

$$= \lim_{T\to\infty} f(T) - f(0) = \lim_{t\to\infty} f(t) - f(0) \qquad (2)$$

From (1) & (2) $\lim_{s\to 0} sF(s) - f(0) = \lim_{t\to\infty} f(t) - f(0)$

- **Conditions:** The function $f(t)$ and $\mathrm{d}f/\mathrm{d}t$ must possess Laplace transforms and $f(t)$ must approach a constant value as $t \to \infty$

Properties of Laplace Transform

- **Initial value theorem**

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$$

$f(t)$ is of exponential order $\Rightarrow$ $\lim_{t\to\infty} e^{-\sigma t}|f(t)| = 0$

$$\lim_{s \to \infty} \int_0^\infty \left[\frac{d}{dt} f(t)\right] e^{-st} dt = 0 = \lim_{s \to \infty} sF(s) - f(0)$$

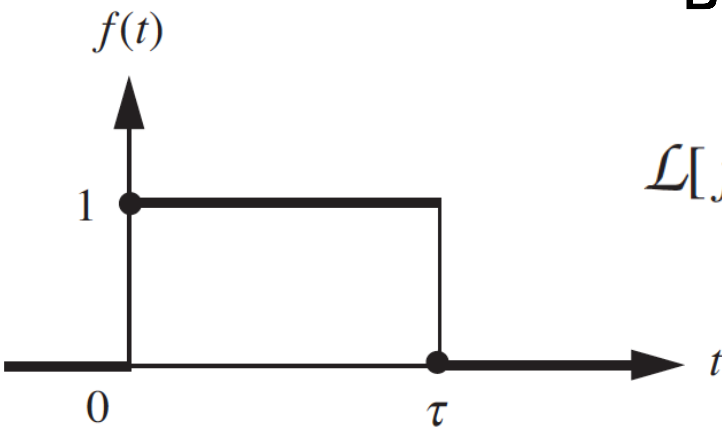
Laplace Transform Table

$\boldsymbol{x(t)}$	$\boldsymbol{X(s)}$
$\delta(t)$	1
$\varepsilon(t)$	$1/s$
$\varepsilon(t)t^n$	$n!/s^{n+1}$
$\varepsilon(t)e^{-\alpha t}$	$1/s+\alpha$
$\varepsilon(t)t^n e^{-\alpha t}$	$n!/(s+\alpha)^{n+1}$
$\varepsilon(t)\cos(\omega t)$	$s/s^2+\omega^2$
$\varepsilon(t)\sin(\omega t)$	$\omega/s^2+\omega^2$
$\varepsilon(t)e^{-\alpha t}\cos(\omega t)$	$(s+\alpha)/(s+\alpha)^2+\omega^2$
$\varepsilon(t)e^{-\alpha t}\sin(\omega t)$	$\omega/(s+\alpha)^2+\omega^2$

Laplace Transform Table

$\boldsymbol{x(t)}$	$\boldsymbol{X(s)}$	
$e^{-\alpha t} f(t)$	$F(s+\alpha)$	
$\varepsilon(t-\tau) f(t-\tau)$	$e^{-s\tau} F(s)$	
$f(t) * g(t)$	$F(s) G(s)$	
$\dfrac{d^n}{dt^n} f(t)$	$s^n F(s) - \sum_{k=1}^{n} s^{n-k} g_{k-1}$	$g_{k-1} = \left. \dfrac{d^{k-1} f}{dt^{k-1}} \right\vert t = 0$
$\int_0^t f(\tau) d\tau$	$\dfrac{F(s)}{s}$	
$t^n f(t)$	$(-1)^n \dfrac{d^n}{ds^n} F(s)$	

Example 1: Decomposition



- **Direct Integration**

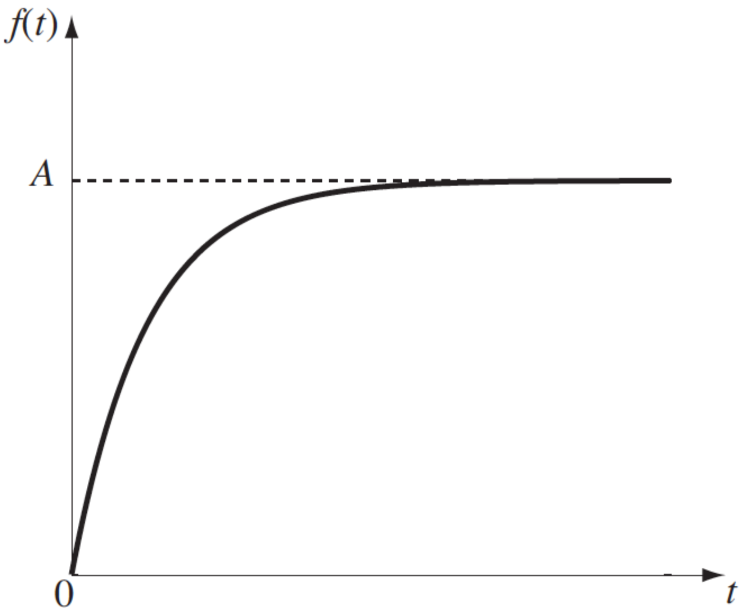
$$\mathcal{L}[f(t)] = \int_0^\tau 1e^{-st}\mathrm{d}t + \int_\tau^\infty 0e^{-st}\mathrm{d}t = \frac{1}{s}(1 - e^{-s\tau})$$

- **Combination of known signals**

$$f(t) = \varepsilon(t) - \varepsilon(t-\tau) \qquad \mathcal{L}[f(t)] = \mathcal{L}[\varepsilon(t)] - \mathcal{L}[\varepsilon(t-\tau)]$$

$$\mathcal{L}[f(t)] = \mathcal{L}[\varepsilon(t)] - e^{-s\tau}\mathcal{L}[\varepsilon(t)] = (1 - e^{-s\tau})\mathcal{L}[\varepsilon(t)] = (1 - e^{-s\tau})\frac{1}{s}$$

Example 2: Exponentials and Final Value



$$f(t) = \varepsilon(t)A[1 - e^{-\alpha t}] \qquad \alpha > 0$$

$$F(s) = \frac{A}{s} - \frac{A}{s+\alpha} = \frac{A\alpha}{s(s+\alpha)}$$

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} [sF(s)] = \lim_{s \to 0} \left[\frac{A\alpha}{s+\alpha}\right] = A$$

Example 3: Time shift

$$S(t) = \begin{cases} \sin t & t \geq 3 \\ 0 & t < 3 \end{cases}$$

$$S(t) = \varepsilon(t-3)\sin(t)$$

$$\sin(t) = \sin(t-3+3) = \sin(t-3)\cos(3) + \cos(t-3)\sin(3)$$

$$\mathcal{L}\{S(t)\} = \mathcal{L}\{\varepsilon(t-3)\sin(t-3)\}\cos(3) + \mathcal{L}\{\varepsilon(t-3)\cos(t-3)\}\sin(3)$$

$$\mathcal{L}\{S(t)\} = e^{-3s}\cos(3)\frac{1}{s^2+1} + e^{-3s}\sin(3)\frac{s}{s^2+1} = \frac{(\cos(3) + s\sin(3))e^{-3s}}{s^2+1}$$

Example 4: Time Derivative

$$u(t) = \cos(\omega t)\varepsilon(t) \qquad U(s) = \frac{s}{s^2 + \omega^2}$$

$$y(t) = \sin(\omega t)\varepsilon(t) \qquad Y(s) = ?$$

$$\frac{d}{dt}\{\cos(\omega t)\} = -\omega \sin(\omega t)$$

$$\mathcal{L}\left\{\frac{d}{dt}\{\cos(\omega t)\}\varepsilon(t)\right\} = \frac{s^2}{s^2 + \omega^2} - u(0) = -\frac{\omega^2}{s^2 + \omega^2}$$

$$-\omega\, \mathcal{L}\{\sin(\omega t)\, \varepsilon(t)\} = -\frac{\omega^2}{s^2 + \omega^2} \quad \Rightarrow \quad \mathcal{L}\{\sin(\omega t)\varepsilon(t)\} = \frac{\omega}{s^2 + \omega^2}$$

Example 5: Inverse Projection

If $F(s) = \dfrac{1}{(s+a)^2}$ then find $f(t)$.

$$\mathcal{L}[t] = \frac{1}{s^2} \quad \text{and} \quad \mathcal{L}[e^{-at} f(t)] = F(s+a)$$

$$\mathcal{L}[e^{-at} t] = \frac{1}{(s+a)^2}$$

Laplace Transform of Convolution

- Assume that $f(t) = g(t) = 0$ for $t < 0$

$$h(t) \equiv \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau = \int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau = f(t) * g(t)$$

$$H(s) = \mathcal{L}\left[\int_0^t f(\tau)g(t-\tau)\,d\tau\right] = \mathcal{L}\left[\int_0^\infty f(\tau)g(t-\tau)\varepsilon(t-\tau)\,d\tau\right]$$

$$= \int_0^\infty e^{-st}\left[\int_0^\infty f(\tau)g(t-\tau)\varepsilon(t-\tau)\,d\tau\right]dt = \int_0^\infty f(\tau)d\tau \int_0^\infty g(t-\tau)\varepsilon(t-\tau)e^{-st}\,dt$$

substitute

$\lambda = t - \tau$

$$H(s) = \int_0^\infty f(\tau)e^{-s\tau}d\tau \int_0^\infty g(\lambda)e^{-s\lambda}\,d\lambda = F(s)G(s)$$

Transfer Function

The output, y(t), of an LTI system with the impulse response g(t) for an input signal, u(t), is given by:

$$y(t) = \int_0^t u(\tau)g(t-\tau)\mathrm{d}\tau = \int_0^t u(t-\tau)g(\tau)\mathrm{d}\tau$$

In the Laplace domain, the relation is given by

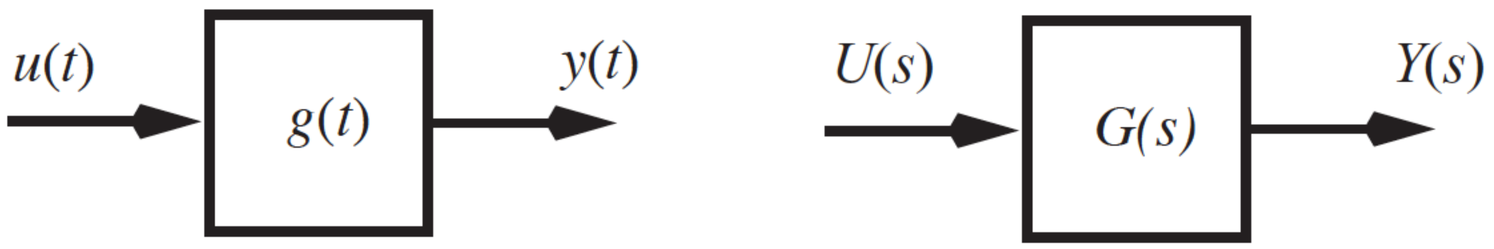
$$\boxed{Y(s) = G(s)U(s)}$$

The Laplace Transform of the impulse response is called **Transfer Function**

$$G(s) \equiv \mathcal{L}\{g(t)\} = \frac{Y(s)}{U(s)}$$

Transfer Function

Input-Output Representation with Transfer Function



Mathematical description of LTI Systems

- Linear State Model: n first order linear differential equations
- Input-output Model: A linear differential equation of order n

State-Space Representation vs Transfer Function

- The dependence on the input signal
- Real vs Complex Variables

Input-Output Model

Linear, time-invariant system

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y^{(1)} + a_0 y$$

$$= b_m u^{(m)} + b_{m-1}u^{(m-1)} + \ldots + b_1 u^{(1)} + b_0 u$$

The transfer function is given by:

$$Y(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} U(s) + \frac{Y_0(s) - U_0(s)}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}$$

Forced response **Free response**

$$Y_0(s) = y_0 s^{n-1} + [y_0^{(1)} + a_{n-1}y_0]s^{n-2} + \ldots + [y_0^{(n-1)} + a_{n-1}y_0^{(n-2)} + \ldots + a_1 y_0]$$

$$U_0(s) = b_m u_0 s^{m-1} + [b_m u_0^{(1)} + b_{m-1}u_0]s^{m-2} + \ldots + [b_m u_0^{(m-1)} + b_{m-1}u_0^{(m-2)} + \ldots + b_1 u_0]$$

Input-Output Model

Linear, time-invariant system that is initially at rest

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y^{(1)} + a_0 y$$

$$= b_m u^{(m)} + b_{m-1}u^{(m-1)} + \ldots + b_1 u^{(1)} + b_0 u$$

The transfer function is given by:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}$$

$m \leq n$

causality

- **Poles** are the roots of the denominator polynomial
- **Zeros** are the roots of the numerator polynomial
- **Order of the system**: degree of denominator polynomial

Example 1

$$\ddot{y}(t) + 2\dot{y}(t) = 2u(t) \qquad y(0) = -1, \quad \dot{y}(0) = 0$$

Reminder:

$$\mathcal{L}\left[\frac{\mathrm{d}}{\mathrm{d}t} f(t)\right] = sF(s) - f(0) \qquad \mathcal{L}\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(t)\right] = s^2F(s) - sf(0) - \frac{\mathrm{d}}{\mathrm{d}t} f(0)$$

$$[s^2Y(s) + s] + 2[sY(s) + 1] = 2U(s)$$

$$Y(s) = \frac{2}{s(s+2)}U(s) - \frac{(s+2)}{s(s+2)} = \frac{2}{s(s+2)}U(s) - \frac{1}{s}$$

Forced response **Free response**

Example 2

$$\dot{x}_1 = -x_1 + x_2 + u \qquad x_1(0) = 0$$

$$\dot{x}_2 = x_1 - 2x_2 \qquad x_2(0) = 0$$

Find

$X_2(s)/U(s)$

$$X_1(s)[s+1] = X_2(s) + U(s)$$

$$X_2(s)[s+2] = X_1(s)$$

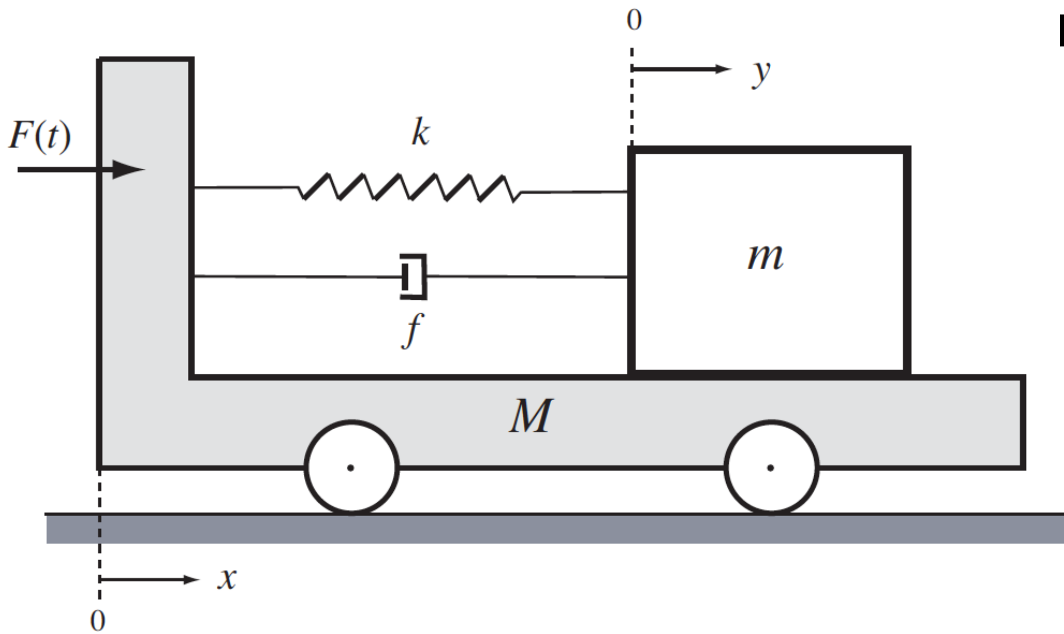
$$X_2(s)[s+2][s+1] = X_2(s) + U(s)$$

$$\frac{X_2(s)}{U(s)} = \frac{1}{s^2 + 3s + 1}$$

Forced response

Example

Initially at rest



$$M\ddot{x} = F + ky + f\dot{y}$$

$$m(\ddot{x} + \ddot{y}) = -ky - f\dot{y}$$

Example

$$M\ddot{x} = F + ky + f\dot{y}$$

$$m(\ddot{x} + \ddot{y}) = -ky - f\dot{y}$$

$$Ms^2X(s) = F(s) + (fs + k)Y(s) \;\Rightarrow\; X(s) = \frac{F(s) + (fs + k)Y(s)}{Ms^2}$$

$$m[s^2X(s) + s^2Y(s)] = -(fs + k)Y(s)$$

$$\frac{Y(s)}{F(s)} = -\frac{1}{Ms^2 + (M/m + 1)fs + (M/m + 1)k}$$

$$\frac{X(s)}{F(s)} = \frac{s^2 + (f/m)s + k/m}{s^2[Ms^2 + (M/m + 1)fs + (M/m + 1)k]}$$

Linear State Model

Linear, time invariant system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

Laplace Transform

$$sX(s) - x_0 = AX(s) + BU(s)$$

$$Y(s) = CX(s) + DU(s)$$

$$\Downarrow$$

$$X(s) = (sI - A)^{-1}BU(s) + (sI - A)^{-1}x_0$$

$$Y(s) = [C(sI - A)^{-1}B + D]U(s) + C(sI - A)^{-1}x_0$$

Linear State Model

Linear, time invariant system that is initially at rest

Transfer Matrix

$$G(s) = C(sI - A)^{-1}B + D$$

$G(s)$: $q \times p$ $\quad$ C: $q \times n$ $\quad$ $(sI - A)^{-1}$: $n \times n$ $\quad$ B: $n \times p$ $\quad$ D: $q \times p$

Example

$$\dot{x}(t) = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$

$$y(t) = (1 \;\; 0)\, x(t)$$

$$G(s) = C(sI - A)^{-1}B + D$$

$$G(s) = [1 \quad 0] \begin{bmatrix} s & -a \\ a & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [1 \quad 0] \begin{bmatrix} \dfrac{s}{s^2 + a^2} & \dfrac{a}{s^2 + a^2} \\ \dfrac{-a}{s^2 + a^2} & \dfrac{s}{s^2 + a^2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{a}{s^2 + a^2}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example

$$\dot{x}(t) = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$

$$y(t) = (1 \;\; 0) x(t)$$

$$\dot{x}_1(t) = a x_2(t) \overset{\mathcal{L}}{\rightarrow} s X_1(s) = a X_2(s)$$

$$\dot{x}_2(t) = -a x_1(t) + u(t) \overset{\mathcal{L}}{\rightarrow} s X_2(s) = -a X_1(s) + U(s)$$

$$y(t) = x_1(t) \overset{\mathcal{L}}{\rightarrow} Y(s) = X_1(s)$$

$$\Rightarrow G(s) = \frac{Y(s)}{U(s)} = \frac{a}{s^2 + a^2}$$